Available online at www.sciencedirect.com
SCIENCE DIRECT®

ELSEVIER

Journal of Geometry and Physics 56 (2006) 1283–1293

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Separation property for Schrödinger operators on Riemannian manifolds

Ognjen Milatovic*

Department of Mathematics and Statistics, University of North Florida, Jacksonville, FL 32224, USA

Received 3 June 2005; received in revised form 3 June 2005; accepted 8 July 2005
Available online 10 August 2005

Abstract

We consider a Schrödinger differential expression $L = \Delta_M + q$ on a complete Riemannian manifold (M, g) with metric g, where Δ_M is the scalar Laplacian on M and $q \geq 0$ is a locally square integrable function on M. In the terminology of Everitt and Giertz, the differential expression L is said to be separated in $L^2(M)$ if for all $u \in L^2(M)$ such that $Lu \in L^2(M)$, we have $qu \in L^2(M)$. We give sufficient conditions for L to be separated in $L^2(M)$.

MSC: 58J50

Keywords: Riemannian manifold; Schrödinger operator; Separation

1. Introduction and the main result

1.1. The setting

Let (M, g) be a Riemannian manifold without boundary (i.e. M is a C^∞-manifold without boundary, (g_{jk}) is a Riemannian metric on M) and $\dim M = n$. We will assume that M is connected. We will also assume that we are given a positive smooth measure $d\mu$, i.e. in any

* Tel.: +1 904 620 1745.
E-mail address: omilatov@unf.edu.

0393-0440/$ – see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2005.07.001

local coordinates $x^1, x^2, \dots, x^n$ there exists a strictly positive C^∞-density $\rho(x)$ such that $d\mu = \rho(x) dx^1 dx^2, \dots, dx^n$.

In the sequel, $L^2(M)$ is the space of complex-valued square integrable functions on M with the inner product:

$$(u, v) = \int_M (u\bar{v})\, d\mu, \tag{1}$$

and $\|\cdot\|$ is the norm in $L^2(M)$ corresponding to the inner product (1).

We use the notation $L^2(\Lambda^1 T^* M)$ for the space of complex-valued square integrable 1-forms on M with the inner product:

$$(\omega, \psi)_{L^2(\Lambda^1 T^* M)} = \int_M \langle \omega, \bar{\psi} \rangle d\mu, \tag{2}$$

where for 1-forms $\omega = \omega_j\, dx^j$ and $\psi = \psi_k\, dx^k$, we define

$$\langle \omega, \psi \rangle := g^{jk} \omega_j \psi_k,$$

where (g^{jk}) is the inverse matrix to (g_{jk}), and

$$\bar{\psi} = \bar{\psi}_k\, dx^k.$$

(Above we used the standard Einstein summation convention.)

The notation $\|\cdot\|_{L^2(\Lambda^1 T^* M)}$ stands for the norm in $L^2(\Lambda^1 T^* M)$ corresponding to the inner product (2).

In what follows, by $C^\infty(M)$ we denote the space of smooth functions on M, by $C_c^\infty(M)$—the space of smooth compactly supported functions on M, by $\Omega^1(M)$—the space of smooth 1-forms on M and by $\Omega_c^1(M)$—the space of smooth compactly supported 1-forms on M.

In the sequel, the operator $d : C^\infty(M) \to \Omega^1(M)$ is the standard differential, and

$$d^* : \Omega^1(M) \to C^\infty(M)$$

is the formal adjoint of d defined by the identity:

$$(du, \omega)_{L^2(\Lambda^1 T^* M)} = (u, d^*\omega), \qquad u \in C_c^\infty(M), \qquad \omega \in \Omega^1(M).$$

By $\Delta_M := d^* d$ we will denote the scalar Laplacian on M.

We consider a Schrödinger-type differential expression:

$$L = \Delta_M + q, \tag{3}$$

where $q \in L^2_{\mathrm{loc}}(M)$ is a real-valued function.

1.2. The set D_1

Let L be as in (3). In the sequel, we will use the notation:

$$D_1 := \{u \in L^2(M) : Lu \in L^2(M)\}, \tag{4}$$

where Lu is understood in the sense of distributions.

Remark 1. In general, it is not true that for all $u \in D_1$ we have $\Delta_M u \in L^2(M)$ and $qu \in L^2(M)$ separately.

Using the terminology of Everitt and Giertz [4], we will say that the differential expression $L = \Delta_M + q$ is separated in $L^2(M)$ when the following statement holds true:

$$\text{for all } u \in D_1, \text{ we have } qu \in L^2(M).$$

We will give sufficient conditions for L to be separated in $L^2(M)$.

First, we make assumptions on q.

Assumption (A1). Assume that there exists a function $0 \le V \in C^1(M)$ such that

$$V(x) \le q(x) \le cV(x) \tag{5}$$

and

$$|\mathrm{d}V(x)| \le \sigma V^{3/2}(x), \qquad \text{for all } x \in M, \tag{6}$$

where $c > 0$ and $0 \le \sigma < 2$ are constants.

In (6), the notation $|\mathrm{d}V(x)|$ denotes the norm of $\mathrm{d}V(x) \in T_x^* M$ with respect to the inner product in $T_x^* M$ induced by the metric g.

Remark 2. From (5) it follows that $0 \le q \in L^\infty_{\text{loc}}(M)$.

We now state the main result.

Theorem 3. *Assume that (M, g) is a connected C^∞-Riemannian manifold without boundary, with metric g and a positive smooth measure $\mathrm{d}\mu$. Additionally, assume that (M, g) is complete. Assume that q satisfies the Assumption (A1). Then*

$$\|\Delta_M u\| + \|qu\| \le C(\|Lu\| + \|u\|), \qquad \text{for all } u \in D_1, \tag{7}$$

where $C \ge 0$ is a constant (independent of u).

The following corollary is an immediate consequence of Theorem 3.

Corollary 4. *Under the hypotheses of* Theorem 3, *the differential expression L is separated in $L^2(M)$.*

Remark 5. Theorem 3 extends a result of Boimatov [1, Theorem 4]concerning the separation property for the Schrödinger operator $-\Delta + q$ in $L^2(\mathbb{R}^n)$, where Δ is the standard Laplacian on $\mathbb{R}^n$ with standard metric and measure and $0 \le q \in C^1(\mathbb{R}^n)$. The problem of separation of differential expressions in $L^2(\mathbb{R}^n)$ has been studied by many authors; see, for instance, [1,4] and references therein.

2. Proof of Theorem 3

2.1. Differential expression L_V

Let $0 \leq V \in C^1(M)$. In the sequel, by L_V we will denote the differential expression $L_V = \Delta_M + V$.

In the two preliminary lemmas, we will adopt the scheme of Boimatov [1] and Everitt and Giertz [4] to our context. In the proof of Theorem 3, we use the positivity preserving property of resolvents of self-adjoint closures of $L_V|_{C_c^\infty(M)}$ and $L|_{C_c^\infty(M)}$.

Lemma 6. *Assume that* (M, g) *is a connected* C^∞*-Riemannian manifold without boundary, with metric g and a positive smooth measure* $\mathrm{d}\mu$. *Assume that* $0 \leq V \in C^1(M)$ *satisfies* (6) *with* $\sigma \in [0, 2)$. *Then the following inequalities hold*:

$$\|\Delta_M u\| + \|Vu\| \leq \tilde{C}\|L_V u\|, \qquad \text{for all } u \in C_c^\infty(M), \tag{8}$$

and

$$\|V^{1/2}\mathrm{d}u\|_{L^2(\Lambda^1 T^*M)} \leq \tilde{C}\|L_V u\|, \qquad \text{for all } u \in C_c^\infty(M), \tag{9}$$

where L_V *is as in* Section 2.1 *and* $\tilde{C}$ *is a constant depending on n and* σ.

Proof. We will first prove that the following equality holds for any $\nu > 0$:

$$\|L_V u\|^2 = \|Vu\|^2 + \nu\|\Delta_M u\|^2 + (1+\nu)\mathrm{Re}(Vu, \Delta_M u) + (1-\nu)\mathrm{Re}(\Delta_M u, L_V u),$$
$$\text{for all } u \in C_c^\infty(M). \tag{10}$$

Let $\nu > 0$ be arbitrary. By the definition of L_V, for all $u \in C_c^\infty(M)$ we have

$$\begin{aligned}\|L_V u\|^2 &= \|Vu\|^2 + \|\Delta_M u\|^2 + 2\mathrm{Re}(\Delta_M u, Vu)\\ &= \|Vu\|^2 + \nu\|\Delta_M u\|^2 + (1-\nu)\|\Delta_M u\|^2 + 2\mathrm{Re}(\Delta_M u, Vu)\\ &= \|Vu\|^2 + \nu\|\Delta_M u\|^2 + (1-\nu)\mathrm{Re}(\Delta_M u, L_V u - Vu) + 2\mathrm{Re}(\Delta_M u, Vu)\\ &= \|Vu\|^2 + \nu\|\Delta_M u\|^2 + (1-\nu)\mathrm{Re}(\Delta_M u, L_V u) + (1+\nu)\mathrm{Re}(\Delta_M u, Vu),\end{aligned}$$

where $(\cdot,\cdot)$ is as in (1) and $\|\cdot\|$ is the corresponding norm in $L^2(M)$.

Since $u \in C_c^\infty(M)$, using integration by parts and the product rule, we have

$$\begin{aligned}\mathrm{Re}(\Delta_M u, Vu) &= \mathrm{Re}(\mathrm{d}^*\mathrm{d}u, Vu) = \mathrm{Re}(\mathrm{d}u, \mathrm{d}(Vu))_{L^2(\Lambda^1 T^*M)}\\ &= \mathrm{Re}(\mathrm{d}u, (\mathrm{d}V)u + V\,\mathrm{d}u)_{L^2(\Lambda^1 T^*M)}\\ &= \mathrm{Re}(\mathrm{d}u, (\mathrm{d}V)u)_{L^2(\Lambda^1 T^*M)} + (\mathrm{d}u, V\,\mathrm{d}u)_{L^2(\Lambda^1 T^*M)} = (\mathrm{Re}Z) + W,\end{aligned} \tag{11}$$

where

$$Z = \int_M \langle \mathrm{d}u, \bar{u}(\mathrm{d}V)\rangle\,\mathrm{d}\mu \tag{12}$$

and

$$W = (V^{1/2}\,\mathrm{d}u, V^{1/2}\,\mathrm{d}u)_{L^2(\Lambda^1 T^*M)}. \tag{13}$$

From (11) we get

$$(1+\nu)\mathrm{Re}(\Delta_M u, Vu) = (1+\nu)\mathrm{Re}Z + (1+\nu)W \geq -(1+\nu)|Z| + (1+\nu)W. \tag{14}$$

We will now estimate $|Z|$, where Z is as in (12). Using the Cauchy-Schwarz inequality and the inequality

$$2ab \leq ka^2 + k^{-1}b^2, \tag{15}$$

where a, b and k are positive real numbers, we get for any $\delta > 0$:

$$\begin{aligned}|Z| &\leq \int_M |\mathrm{d}u||\mathrm{d}V||u|\,\mathrm{d}\mu \leq \sigma \int_M V^{3/2}|u||\mathrm{d}u|\,\mathrm{d}\mu = \sigma \int_M |V^{1/2}\,\mathrm{d}u||Vu|\,\mathrm{d}\mu \\ &\leq \frac{\nu\delta}{2}\|V^{1/2}\,\mathrm{d}u\|^2_{L^2(\Lambda^1 T^*M)} + \frac{\sigma^2}{2\nu\delta}\|Vu\|^2.\end{aligned} \tag{16}$$

Here, for $\xi \in T_x^*M$, the notation $|\xi|$ denotes the norm of ξ with respect to the inner product in T_x^*M induced by the metric g. In the second inequality in (16), we used (6), and in the third inequality in (16) we used (15).

Using (15), for any $\alpha > 0$ we get

$$|\mathrm{Re}(\Delta_M u, L_V u)| \leq |(\Delta_M u, L_V u)| \leq \frac{\alpha}{2}\|\Delta_M u\|^2 + \frac{1}{2\alpha}\|L_V u\|^2. \tag{17}$$

Combining (10), (14), (16) and (17) we obtain

$$\begin{aligned}\|L_V u\|^2 \geq\; &\|Vu\|^2 + \nu\|\Delta_M u\|^2 - \frac{(1+\nu)\nu\delta}{2}\|V^{1/2}\,\mathrm{d}u\|^2_{L^2(\Lambda^1 T^*M)} \\ &- \frac{(1+\nu)\sigma^2}{2\nu\delta}\|Vu\|^2 + (1+\nu)\|V^{1/2}\,\mathrm{d}u\|^2_{L^2(\Lambda^1 T^*M)} \\ &- \frac{|1-\nu|\alpha}{2}\|\Delta_M u\|^2 - \frac{|1-\nu|}{2\alpha}\|L_V u\|^2.\end{aligned} \tag{18}$$

From (18) we get

$$\begin{aligned}\left(1 + \frac{|1-\nu|}{2\alpha}\right)\|L_V u\|^2 \geq &\left(1 - \frac{(1+\nu)\sigma^2}{2\nu\delta}\right)\|Vu\|^2 + \left(\nu - \frac{|1-\nu|\alpha}{2}\right)\|\Delta_M u\|^2 \\ &+ \left((1+\nu) - \frac{(1+\nu)\nu\delta}{2}\right)\|V^{1/2}\,\mathrm{d}u\|^2_{L^2(\Lambda^1 T^*M)}.\end{aligned} \tag{19}$$

The inequalities (8) and (9) will immediately follow from (19) if

$$|1-\nu| < \frac{2\nu}{\alpha}, \qquad \nu\delta < 2, \quad \text{and} \quad (1+\nu)\sigma^2 < 2\nu\delta. \tag{20}$$

Since, by hypothesis, $0 \leq \sigma < 2$, there exist numbers $\nu > 0$, $\alpha > 0$ and $\delta > 0$ such that the inequalities (20) hold.

This concludes the proof of the lemma. □

In the sequel, we will also use the following terms and notations.

2.2. *Minimal operators S and T*

Let L be as in (3) and let L_V be as in Section 2.1. We define the minimal operators S and T in $L^2(M)$ associated to L and L_V by the formulas $Su = Lu$ and $Tu = L_V u$ with the domains $\mathrm{Dom}(S) = \mathrm{Dom}(T) = C_c^\infty(M)$. Since S and T are symmetric operators, it follows that S and T is closable; see, for example, Section V.3.3 in [6]. In what follows, we will denote by $\tilde{S}$ and $\tilde{T}$ the closures in $L^2(M)$ of the operators S and T, respectively.

2.3. *Maximal operators H and K*

Let L be as in (3). We define the maximal operator H in $L^2(M)$ associated to L by the formula $Hu = S^*u$, where S^* is the adjoint of the operator S in $L^2(M)$. In the case when $q \in L^2_{\mathrm{loc}}(M)$ is real-valued, it is well-known that $\mathrm{Dom}(H) = D_1$, where D_1 is as in (4). Let L_V be as in Section 2.1. We define the maximal operator K in $L^2(M)$ associated to L_V by the formula $Ku = T^*u$, and we have $\mathrm{Dom}(K) = \{u \in L^2(M) : L_V u \in L^2(M)\}$.

Remark 7. By Lemma 5.1 from [9] it follows that $\mathrm{Dom}(K) \subset W^{2,2}_{\mathrm{loc}}(M)$.

2.4. *Essential self-adjointness of S and T*

If (M, g) is a complete Riemannian manifold with metric g and positive smooth measure $\mathrm{d}\mu$ and if $0 \leq q \in L^2_{\mathrm{loc}}(M)$, then by Theorem 1.1 in [9] (or by Corollary 2.9 in [2]) the operator S is essentially self-adjoint in $L^2(M)$. In this case, we have $\tilde{S} = S^*$; see, for example, Section V.3.3 in [6]. In particular, since $0 \leq V \in C^1(M) \subset L^2_{\mathrm{loc}}(M)$, it follows that the operator T is essentially self-adjoint in $L^2(M)$.

Lemma 8. *Assume that (M, g) is a connected C^∞-Riemannian manifold without boundary, with metric g and a positive smooth measure $\mathrm{d}\mu$. Additionally, assume that (M, g) is complete. Assume that $0 \leq V \in C^1(M)$ satisfies (6). Then the inequalities (8) and (9) hold for all $u \in \mathrm{Dom}(K)$, where K is as in* Section 2.3.

Proof. Under the hypotheses of this lemma, by Section 2.4 it follows that T is essentially self-adjoint and $K = T^* = \tilde{T}$. In particular, $\mathrm{Dom}(\tilde{T}) = \mathrm{Dom}(K)$.

Let $u \in \mathrm{Dom}(K)$. Then there exists a sequence $\{u_k\}$ in $C_c^\infty(M)$ such that

$$u_k \to u \quad \text{and} \quad L_V u_k \to \tilde{T}u \quad \text{in} \quad L^2(M), \quad \text{as } k \to \infty.$$

Since by Lemma 6 the sequence $\{u_k\}$ satisfies (8) and (9), it follows that the sequences $\{Vu_k\}$ and $\{\Delta_M u_k\}$ are Cauchy sequences in $L^2(M)$, and $\{V^{1/2}\mathrm{d}u_k\}$ is a Cauchy sequence in $L^2(\Lambda^1 T^* M)$.

We will first show that

$$Vu_k \to Vu \quad \text{in} \quad L^2(M), \quad \text{as} \quad k \to \infty. \tag{21}$$

Since $\{Vu_k\}$ is a Cauchy sequence in $L^2(M)$, it follows that Vu_k converges to $s \in L^2(M)$. Let ϕ be an arbitrary element of $C_c^\infty(M)$. Then

$$0 = (Vu_k, \phi) - (u_k, V\phi) \to (s, \phi) - (u, V\phi) = (s - Vu, \phi), \tag{22}$$

where $(\cdot, \cdot)$ is as in (1).

Since $C_c^\infty(M)$ is dense in $L^2(M)$, we get $s = Vu$, and (21) is proven.

If (M, g) is complete, it is well-known that Δ_M is essentially self-adjoint on $C_c^\infty(M)$, and we have the following equality:

$$(\Delta_M|_{C_c^\infty(M)})^\sim = \Delta_{M,\max},$$

where $\Delta_{M,\max}u := \Delta_M u$ with the domain

$$\mathrm{Dom}(\Delta_{M,\max}) = \{u \in L^2(M) : \Delta_M u \in L^2(M)\};$$

see, for example, Theorem 3.5 in [3].

Since $u_k \to u$ in $L^2(M)$ and since $\{\Delta_M u_k\}$ is a Cauchy sequence in $L^2(M)$, by the definition of $(\Delta_M|_{C_c^\infty(M)})^\sim$ it follows that $u \in \mathrm{Dom}((\Delta_M|_{C_c^\infty(M)})^\sim)$. Since $(\Delta_M|_{C_c^\infty(M)})^\sim = \Delta_{M,\max}u$, we have

$$\Delta_M u_k \to \Delta_M u \quad \text{in} \quad L^2(M), \quad \text{as} \quad k \to \infty. \tag{23}$$

Since $\{\Delta_M u_k\}$ and $\{u_k\}$ are Cauchy sequences in $L^2(M)$ and since

$$\|\mathrm{d}u_k\|^2_{L^2(\Lambda^1 T^*M)} = (\mathrm{d}u_k,\ \mathrm{d}u_k)_{L^2(\Lambda^1 T^*M)} = (\Delta_M u_k, u_k) \le \|\Delta_M u_k\| \|u_k\|,$$

it follows that $\{\mathrm{d}u_k\}$ is a Cauchy sequence in $L^2(\Lambda^1 T^*M)$, and, hence, $\mathrm{d}u_k$ converges to some element $\omega \in L^2(\Lambda^1 T^*M)$. Let $\psi \in \Omega_c^1(M)$ be arbitrary. Then, using integration by parts (see, for example, Lemma 8.8 in [2]) and Remark 7, we get

$$\begin{aligned} 0 &= (\mathrm{d}u_k, \psi)_{L^2(\Lambda^1 T^*M)} - (u_k, \mathrm{d}^*\psi) \to (\omega, \psi)_{L^2(\Lambda^1 T^*M)} - (u, \mathrm{d}^*\psi) \\ &= (\omega, \psi)_{L^2(\Lambda^1 T^*M)} - (\mathrm{d}u, \psi)_{L^2(\Lambda^1 T^*M)}, \end{aligned} \tag{24}$$

where $(\cdot, \cdot)$ is the inner product in $L^2(M)$.

From (24) we get $\mathrm{d}u = \omega \in L^2(\Lambda^1 T^*M)$, and, hence,

$$\mathrm{d}u_k \to \mathrm{d}u, \quad \text{in} \quad L^2(\Lambda^1 T^*M), \quad \text{as} \quad k \to \infty. \tag{25}$$

Since $\{V^{1/2}\,\mathrm{d}u_k\}$ is a Cauchy sequence in $L^2(\Lambda^1 T^*M)$, using (25) and the same argument as in (22), we obtain

$$V^{1/2}\,\mathrm{d}u_k \to V^{1/2}\,\mathrm{d}u, \quad \text{in} \quad L^2(\Lambda^1 T^*M), \quad \text{as} \quad k \to \infty. \tag{26}$$

Using (21), (23), (26) and taking limits as $k \to \infty$ in all terms in (8) and (9) (with u replaced by u_k), shows that (8) and (9) hold for all $u \in \mathrm{Dom}(\tilde{T}) = \mathrm{Dom}(K)$. This concludes the proof of the lemma. □

2.5. Operators R_1 and R_2

Let S and T be as in Section 2.2 and let H and K be as in Section 2.3. By Section 2.4 the operators $H = \tilde{S}$ and $K = \tilde{T}$ are non-negative self-adjoint in $L^2(M)$. Thus,

$$R_1 := (H + 1)^{-1} : L^2(M) \to L^2(M),$$
$$R_2 := (K + 1)^{-1} : L^2(M) \to L^2(M) \tag{27}$$

are bounded linear operators.

2.6. Positivity preserving property

For the following definition, see, for example, the definition below the formulation of Theorem X.30 in [8].

Let (X, μ) be a measure space. A bounded linear operator $A : L^2(X, \mu) \to L^2(X, \mu)$ is said to be *positivity preserving* if for every $u \in L^2(X, \mu)$ such that $u \geq 0$ a.e. on X, we have $Au \geq 0$ a.e. on X.

Remark 9. Let $A : L^2(X, \mu) \to L^2(X, \mu)$ be a positivity preserving bounded linear operator. Then the following inequality holds for all $u \in L^2(X, \mu)$:

$$|(Au)(x)| \leq A|u(x)|, \quad \text{a.e. on} \quad X, \tag{28}$$

where $|\cdot|$ denotes the absolute value of a complex number. For the proof of (28), see the proof of the inequality (X.103) in [8].

In the sequel, we will use the following proposition.

Proposition 10. *Assume that (M, g) is a (not necessarily complete) C^∞-Riemannian manifold without boundary. Assume that M is connected and oriented. Assume that $Q_0 \in L^2_{\text{loc}}(M)$ is real-valued. Additionally, assume that*

$$((\Delta_M + Q_0)u, u) \geq 0, \quad \textit{forall } u \in C_c^\infty(M).$$

Let S_0 be the Friedrichs extension of $(\Delta_M + Q_0)|_{C_c^\infty(M)}$. Assume that λ is a positive real number. Then the operator $(S_0 + \lambda)^{-1}$ is positivity preserving.

Remark 11. For the proof of Proposition 10, which is based on Kato's inequality technique on Riemannian manifolds, see the proof of Proposition 2.13 in [7]. Proposition 10 is an extension to Riemannian manifolds of Lemma 2 from Goelden [5]. For more on Kato's inequality technique on Riemannian manifolds and its application to essential self-adjointness of Schrödinger-type operators, see [2] and references there.

From Proposition 10 we obtain the following corollary.

Corollary 12. *Assume that* (M, g) *is a complete* C^∞*-Riemannian manifold without boundary. Assume that* M *is connected and oriented. Assume that* $0 \le q \in L^2_{\text{loc}}(M)$ *and* $0 \le V \in C^1(M)$. *Let* R_1 *and* R_2 *be as in* (27). *Then the operators* R_1 *and* R_2 *are positivity preserving*.

Proof. Since (M, g) is complete, by Section 2.4 it follows that S and T are non-negative essentially self-adjoint operators in $L^2(M)$. Thus, the Friedrichs extensions of $(\Delta_M + q)|_{C_c^\infty(M)}$ and $(\Delta_M + V)|_{C_c^\infty(M)}$ are H and K, respectively. By Proposition 10 it follows that R_1 and R_2 are positivity preserving operators in $L^2(M)$. □

Proof of Theorem 3. By Lemma 8 it follows that

$$\|Vu\| \le \tilde{C}\|Ku\|, \quad \text{for all } u \in \text{Dom}(K). \tag{29}$$

Let R_2 be as in (27) and let $V : L^2(M) \to L^2(M)$ denote the maximal multiplication operator corresponding to the function V.

From (29) it follows that $VR_2 : L^2(M) \to L^2(M)$ is a bounded linear operator.

Let $q : L^2(M) \to L^2(M)$ denote the maximal multiplication operator corresponding to the function q. Since $VR_2 : L^2(M) \to L^2(M)$ is a bounded linear operator, by (5) it follows that $qR_2 : L^2(M) \to L^2(M)$ is a bounded linear operator.

We will now show that qR_1 is a bounded linear operator: $L^2(M) \to L^2(M)$.

First, we will show that

$$qR_1 f \le qR_2 f, \quad \text{for all } 0 \le f \in L^2(M). \tag{30}$$

Here, the inequality is understood in a pointwise sense. (Since $(R_1 f) \in D_1 \subset L^2(M)$ and since $q \in L^2_{\text{loc}}(M)$, it follows that $(qR_1 f) \in L^1_{\text{loc}}(M)$ is a function.)

Let $0 \le f \in L^2(M)$ be arbitrary. Since

$$R_2 f \in \text{Dom}(K) = \{z \in L^2(M) : L_V z \in L^2(M)\},$$

by Lemma 8 we have $(\Delta_M(R_2 f)) \in L^2(M)$ and $(VR_2 f) \in L^2(M)$. Since $qR_2 : L^2(M) \to L^2(M)$ is a bounded linear operator, we get $(qR_2 f) \in L^2(M)$. Thus $R_2 f \in D_1$, and, hence,

$$qR_2 f = qR_1(H + 1)R_2 f. \tag{31}$$

By the definition of R_2 have

$$qR_1 f = qR_1(K + 1)R_2 f. \tag{32}$$

From (31) and (32) we have

$$qR_2 f - qR_1 f = qR_1((H + 1)R_2 f - (K + 1)R_2 f) = qR_1((q - V)R_2 f). \tag{33}$$

Since R_2 is positivity preserving in $L^2(M)$, we have $R_2 f \ge 0$. Since VR_2 and qR_2 are bounded linear operators $L^2(M) \to L^2(M)$ and since $q - V \ge 0$, we have

$$0 \le (q - V)R_2 f \in L^2(M). \tag{34}$$

Since R_1 is positivity preserving in $L^2(M)$, from (34) we get

$$R_1((q-V)R_2 f) \geq 0. \tag{35}$$

Since $q \geq 0$, from (33) and (35) we get

$$qR_2 f - qR_1 f = qR_1((q-V)R_2 f) \geq 0,$$

and (30) is proven.

Let $w \in L^2(M)$ be arbitrary. Since $R_1 : L^2(M) \to L^2(M)$ is a positivity preserving bounded linear operator, by (28) we have the following pointwise inequality:

$$|R_1 w| \leq R_1 |w|. \tag{36}$$

Since $q \geq 0$, from (36) and (30) we obtain the following pointwise inequality:

$$|qR_1 w| = q|R_1 w| \leq qR_1|w| \leq qR_2|w|. \tag{37}$$

Since $qR_2 : L^2(M) \to L^2(M)$ is a bounded linear operator, from (37) it follows that

$$\|qR_1 w\| \leq \|qR_2|w|\| \leq C_3\|w\|, \quad \text{for all } w \in L^2(M),$$

where $C_3 \geq 0$ is a constant.

Hence, qR_1 is a bounded linear operator: $L^2(M) \to L^2(M)$.

Let $u \in D_1$ be arbitrary. Then $qu = qR_1(u + Lu)$, where $Lu = \Delta_M u + qu$.

Since $qR_1 : L^2(M) \to L^2(M)$ is a bounded linear operator, we have

$$\|qu\| \leq C_1(\|u\| + \|Lu\|), \quad \text{for all } u \in D_1, \tag{38}$$

where $C_1 \geq 0$ is a constant.

Using (38) we get

$$\|\Delta_M u\| = \|Lu - qu\| \leq \|Lu\| + \|qu\| \leq C_2(\|u\| + \|Lu\|), \quad \text{for all } u \in D_1, \tag{39}$$

where $C_2 \geq 0$ is a constant.

From (38) and (39) we obtain (7). This concludes the proof of the Theorem. □

References

[1] K.Kh. Boimatov, Coercive estimates and separation for second order elliptic differential equations, Sov. Math. Dokl. 38 (1) (1989) 157–160.

[2] M. Braverman, O. Milatovic, M. Shubin, Essential self-adjointness of Schrödinger type operators on manifolds, Russ. Math. Surv. 57 (4) (2002) 641–692.

[3] J. Eichhorn, Elliptic differential operators on noncompact manifolds, in: Proceedings of the Seminar Analysis of the Karl-Weierstrass-Institute of Mathematics, Teubner, Leipzig, Berlin, 1986/87, Teubner-Texte Math. 106, 1988, 4–169.

[4] W.N. Everitt, M. Giertz, Inequalities and separation for Schrödinger type operators in $L^2(\mathbb{R}^n)$, Proc. Royal Soc. Edinb. 79A (1977) 257–265.

[5] H.-W. Goelden, On non-degeneracy of the ground state of Schrödinger operators, Math. Z. 155 (1977) 239–247.
[6] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, New York, 1980.
[7] O. Milatovic, The form sum and the Friedrichs extension of Schrödinger-type operators on Riemannian manifolds, Proc. Am. Math. Soc. 132 (1) (2004) 147–156.
[8] M. Reed, B. Simon, Methods of Modern Mathematical Physics I, II: Functional Analysis. Fourier Analysis, Self-adjointness, Academic Press, New York, 1975.
[9] M.A. Shubin, Essential self-adjointness for semibounded magnetic Schrödinger operators on non-compact manifolds, J. Funct. Anal. 186 (2001) 92–116.